Deformed preprojective algebras and symplectic reflection algebras for wreath products

Wee Liang Gan and Victor Ginzburg

Abstract

We determine the PBW deformations of the wreath product of a symmetric group with a deformed preprojective algebra of an affine Dynkin quiver. In particular, we show that there is precisely one parameter which does not come from deformation of the preprojective algebra. We prove that the PBW deformation is Morita equivalent to a corresponding symplectic reflection algebra for wreath product.

1 Introduction

1.1 Deformed preprojective algebras were introduced by Crawley-Boevey and Holland in [CBH]. We start by recalling its definition.

Let Q be a quiver, and denote by I the set of vertices of Q. The double $\overline{Q}$ of Q is the quiver obtained from Q by adding a reverse edge $a^* : j \to i$ for each edge $a : i \to j$ in Q. For any edge $a : i \to j$ in $\overline{Q}$, we write its tail $t(a) := i$ and its head $h(a) := j$. Let $B := \bigoplus_{i\in I} \mathbb{C}$, and E the vector space over $\mathbb{C}$ with basis formed by the set of edges $\{a \in \overline{Q}\}$. Thus, E is a B-bimodule and $E = \bigoplus_{i,j\in I} E_{i,j}$, where $E_{i,j}$ is spanned by the edges $a \in \overline{Q}$ with $h(a) = i$ and $t(a) = j$. The path algebra of $\overline{Q}$ is $\mathbb{C}\overline{Q} := T_B E = \bigoplus_{n\geq 0} T^n_B E$, where $T^n_B E = E \otimes_B \cdots \otimes_B E$ is the n-fold tensor product. The trivial path for the vertex i is denoted by e_i, an idempotent in B. Let $r := \sum_{a\in Q}[a, a^*] \in T^2_B E$. For each $i \in I$, let

$$r_i := e_i r e_i = \sum_{\{a\in Q \,|\, h(a)=i\}} a \cdot a^* - \sum_{\{a\in Q \,|\, t(a)=i\}} a^* \cdot a\,.$$

For an element $\lambda \in B$, we will write $\lambda = \sum_{i\in I} \lambda_i e_i$ where $\lambda_i \in \mathbb{C}$.

Definition 1.1.1. For each element $\lambda \in B$, the *deformed preprojective algebra* of Q is the quotient algebra

$$\Pi_\lambda := \frac{\mathbb{C}\overline{Q}}{\langle\langle\, r - \lambda \,\rangle\rangle} = \frac{\mathbb{C}\overline{Q}}{\langle\langle\, r_i - \lambda_i e_i \,\rangle\rangle_{i\in I}}\ ,$$

where $\langle\langle \ldots \rangle\rangle$ is the two-sided ideal generated by the indicated elements.

The algebra Π_0 is called the preprojective algebra of Q. Note that the grading on $\mathbb{C}\overline{Q}$ induces a filtration on Π_λ and there is a natural map $\Pi_0 \longrightarrow \mathtt{gr}\,\Pi_\lambda$. The “PBW” theorem for Π_λ proved in [CBH, Cor. 3.6] says that this map is an isomorphism when Q is an affine Dynkin quiver (of type ADE).

1.2 The main construction of this paper is a one-parameter deformation of the wreath product $\Pi_\lambda^{\otimes n}\#S_n$, where n is an integer greater than 1, the superscript $\otimes n$ means n-fold tensor product over $\mathbb{C}$, and S_n is the symmetric group on n objects.

To state the definition, we will use the following notations. Let n be a positive integer. The element $s_{ij} \in S_n$ is the transposition $i \leftrightarrow j$. Let $\mathsf{B} := B^{\otimes n}$. For any $\ell \in [1,n]$, define the B-bimodules

$$\mathsf{E}_\ell := B^{\otimes(\ell-1)} \otimes E \otimes B^{\otimes(n-\ell)} \qquad \text{and} \qquad \mathsf{E} := \bigoplus_{1\le\ell\le n} \mathsf{E}_\ell\,.$$

The natural inclusions $\mathsf{E}_\ell \hookrightarrow B^{\otimes(\ell-1)} \otimes T_B E \otimes B^{\otimes(n-\ell)} \subset (T_B E)^{\otimes n}$ induce canonical identifications $T_{\mathsf{B}}\mathsf{E}_\ell = B^{\otimes(\ell-1)} \otimes T_B E \otimes B^{\otimes(n-\ell)}$ and a surjective morphism $\Upsilon : T_{\mathsf{B}}\mathsf{E} \longrightarrow (T_B E)^{\otimes n}$. Given two elements $\varepsilon \in \mathsf{E}_\ell$ and $\varepsilon' \in \mathsf{E}_m$ of the form

$$\varepsilon = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes a \otimes \cdots \otimes h(b) \otimes \cdots \otimes e_{i_n}\,, \tag{1.2.1}$$

$$\varepsilon' = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes t(a) \otimes \cdots \otimes b \otimes \cdots \otimes e_{i_n}\,, \tag{1.2.2}$$

where $\ell \neq m$, $a,b \in \overline{Q}$ and $i_1,\dots,i_n \in I$, we define their "commutator"

$$\begin{aligned} \lfloor \varepsilon, \varepsilon' \rfloor \;:=\; & (e_{i_1} \otimes \cdots \otimes a \otimes \cdots \otimes h(b) \otimes \cdots \otimes e_{i_n})(e_{i_1} \otimes \cdots \otimes t(a) \otimes \cdots \otimes b \otimes \cdots \otimes e_{i_n}) \\ & -(e_{i_1} \otimes \cdots \otimes h(a) \otimes \cdots \otimes b \otimes \cdots \otimes e_{i_n})(e_{i_1} \otimes \cdots \otimes a \otimes \cdots \otimes t(b) \otimes \cdots \otimes e_{i_n}). \end{aligned}$$

Note that $\lfloor \varepsilon, \varepsilon' \rfloor$ is an element in $T^2_{\mathsf{B}}\mathsf{E}$. The kernel of Υ is the two-sided ideal generated by all elements of the form $\lfloor \varepsilon, \varepsilon' \rfloor$.

Definition 1.2.3. Let $n > 1$ be an integer. For any $\lambda \in B$ and $\nu \in \mathbb{C}$, define the algebra $\mathsf{A}_{n,\lambda,\nu}$ to be the quotient of $T_{\mathsf{B}}\mathsf{E}\,\#S_n$ by the following relations.

(i) For any $i_1,\dots,i_n \in I$ and $\ell \in [1,n]$:

$$e_{i_1} \otimes \cdots \otimes (r_{i_\ell} - \lambda_{i_\ell} e_{i_\ell}) \otimes \cdots \otimes e_{i_n} = \nu \sum_{\{j\neq\ell \,|\, i_j = i_\ell\}} (e_{i_1} \otimes \cdots \otimes e_{i_\ell} \otimes \cdots \otimes e_{i_n}) s_{j\ell}\,.$$

(ii) For any $\varepsilon, \varepsilon'$ of the form (1.2.1)–(1.2.2):

$$\lfloor \varepsilon, \varepsilon' \rfloor = \begin{cases} \nu(e_{i_1} \otimes \cdots \otimes h(a) \otimes \cdots \otimes t(a) \otimes \cdots \otimes e_{i_n}) s_{\ell m} & \text{if } a = b^* \;\&\; b \in Q\,, \\ -\nu(e_{i_1} \otimes \cdots \otimes h(a) \otimes \cdots \otimes t(a) \otimes \cdots \otimes e_{i_n}) s_{\ell m} & \text{if } b = a^* \;\&\; a \in Q\,, \\ 0 & \text{else}\,. \end{cases}$$

It is easy to see that $\mathsf{A}_{n,\lambda,\nu}$ does not depend on the orientation of Q, cf. [CBH, Lemma 2.2]. Moreover, $\mathsf{A}_{n,\lambda,0} = \Pi_\lambda^{\otimes n}\#S_n$. The grading on $T_{\mathsf{B}}\mathsf{E}$ induces a filtration on $\mathsf{A}_{n,\lambda,\nu}$ and there is a natural map $\Pi_0^{\otimes n}\#S_n = \mathsf{A}_{n,0,0} \longrightarrow \mathtt{gr}\,\mathsf{A}_{n,\lambda,\nu}$. We will prove in §2 that, when Q is affine Dynkin of type ADE, this map is an isomorphism, and any "PBW deformation" of $\Pi_0^{\otimes n}\#S_n$ must be of the form $\mathsf{A}_{n,\lambda,\nu}$.

Remark 1.2.4. Regarding ν as a formal variable, we obtain a one-parameter *formal* deformation of $\Pi_\lambda^{\otimes n}\#S_n$; see [EO].

1.3 The motivation to study $\mathsf{A}_{n,\lambda,\nu}$ comes from [EG], in which Etingof and Ginzburg introduced the so-called symplectic reflection algebras for wreath products, cf. also [GS]. We recall its definition.

Let L be a 2-dimensional vector space equipped with a nondegenerate symplectic form ω_L. Let $V := L^{\oplus n}$ and $\omega := \omega_L^{\oplus n}$. Let Γ be a finite subgroup of $Sp(L)$ and $\boldsymbol{\Gamma}_n := S_n \ltimes \Gamma^n \subset Sp(V)$. Denote by $\mathsf{Z}\Gamma$ the center of the group algebra $\mathbb{C}[\Gamma]$. Given $\gamma \in \Gamma$, write $\gamma_i \in \boldsymbol{\Gamma}_n$ for γ placed in the i-th factor Γ. An element $s \in \boldsymbol{\Gamma}_n$ is called a symplectic reflection if $\mathtt{rk}(\mathtt{Id} - s) = 2$. According to [EG, (11.1)], there are two types of symplectic reflections in $\boldsymbol{\Gamma}_n$:

(S) The elements $s_{ij}\gamma_i\gamma_j^{-1}$, where $i, j \in [1, n]$ and $\gamma \in \Gamma$.

(Γ) The elements γ_i, where $i \in [1, n]$ and $\gamma \in \Gamma \smallsetminus \{1\}$.

The group $\boldsymbol{\Gamma}_n$ acts on the set $\mathcal{S}$ of symplectic reflections by conjugation. The set of elements of type (S) form a single $\boldsymbol{\Gamma}_n$-conjugacy class, while the elements of type (Γ) for γ in each Γ-conjugacy class form a $\boldsymbol{\Gamma}_n$-conjugacy class. Thus, we may identify an $\mathtt{Ad}\,\boldsymbol{\Gamma}_n$-invariant function $c : \mathcal{S} \longrightarrow \mathbb{C} \,:\, s \mapsto c_s$ with an element $k \cdot 1 + \sum_{\gamma \in \Gamma \smallsetminus \{1\}} c'_\gamma \cdot \gamma \in \mathsf{Z}\Gamma$, where k is the value of c on elements of type (S) and c'_γ is the value of c on the elements γ_i of type (Γ). For each $s \in \mathcal{S}$, write ω_s for the bilinear form on V which coincides with ω on $\mathtt{Im}(\mathtt{Id} - s)$ and has $\mathtt{Ker}(\mathtt{Id} - s)$ as its radical.

Definition 1.3.1. For any $t \in \mathbb{C}$ and $c \in \mathsf{Z}\Gamma$, the *symplectic reflection algebra* $\mathsf{H}_{t,c}(\boldsymbol{\Gamma}_n)$ is defined to be the quotient algebra $(TV\#\boldsymbol{\Gamma}_n)/\langle\langle\, [u,v] - \kappa(u,v)\,\rangle\rangle_{u,v\in V}$, where

$$\kappa \,:\, V \otimes V \longrightarrow \mathbb{C}[\boldsymbol{\Gamma}_n] \,:\, (u,v) \mapsto t \cdot \omega(u,v) \cdot 1 + \sum_{s \in \mathcal{S}} c_s \cdot \omega_s(u,v) \cdot s\,.$$

We will construct in §3 a Morita equivalence between $\mathsf{H}_{t,c}(\boldsymbol{\Gamma}_n)$ and $\mathsf{A}_{n,\lambda,\nu}$, where the quiver Q is associated to Γ via the *McKay correspondence.*

Remark 1.3.2. When $\Gamma = \{1\}$, the algebra $\mathsf{H}_{t,c}(\boldsymbol{\Gamma}_n) = \mathsf{H}_{t,k}(S_n)$ is the rational Cherednik algebra of type A_{n-1}, cf. [EG] and Lemma 3.1.1 below. In this case, the quiver Q is the affine Dynkin quiver of type A_0, and we have $\mathsf{A}_{n,\lambda,\nu} = \mathsf{H}_{t,k}(S_n)$ where the parameters $\lambda = t$ and $\nu = k/2$.

2 PBW deformation

2.1 We will define here what we mean by PBW deformations of $\Pi_0^{\otimes n}\#S_n$.

Denote by $\lfloor \mathsf{E}_\ell, \mathsf{E}_m \rfloor$ the sub-B-bimodule of $T^2_{\mathsf{B}}\mathsf{E}$ spanned by all elements of the form $\lfloor \varepsilon, \varepsilon' \rfloor$ with $\varepsilon \in \mathsf{E}_\ell$, $\varepsilon' \in \mathsf{E}_m$. Let R be the sub-B-bimodule of $T^2_B E$ spanned by r_i for $i \in I$, and let

$$\mathsf{R}_\ell := B^{\otimes(\ell-1)} \otimes R \otimes B^{\otimes(n-\ell)}\,, \quad \mathsf{R} := \Big(\bigoplus_{1\le\ell\le n} \mathsf{R}_\ell\Big) \oplus \Big(\bigoplus_{1\le\ell<m\le n} \lfloor \mathsf{E}_\ell, \mathsf{E}_m \rfloor\Big) \subset T^2_{\mathsf{B}}\mathsf{E}\,.$$

Furthermore, let $K := \mathsf{B}\#S_n$ and $M := \mathsf{E}\otimes\mathbb{C}[S_n]$. Note that M is a K-bimodule, where the left action of S_n on M is the diagonal one. We have: $T_{\mathsf{B}}\mathsf{E}\,\#S_n = T_K M$. Let $U := \mathsf{R}\otimes\mathbb{C}[S_n] \subset T_K^2 M$. For any K-bilinear map $\beta : U \longrightarrow K$, define the algebra

$$\mathsf{A}_\beta := \frac{T_{\mathsf{B}}\mathsf{E}\,\#S_n}{\langle\langle\, x-\beta(x)\,\rangle\rangle_{x\in U}} = \frac{T_K M}{\langle\langle\, x-\beta(x)\,\rangle\rangle_{x\in U}}\,.$$

The grading on $T_{\mathsf{B}}\mathsf{E}\,\#S_n$ induces a filtration on A_β and there is a natural map $\Pi_0^{\otimes n}\#S_n = \mathsf{A}_0 \longrightarrow \mathtt{gr}\,\mathsf{A}_\beta$.

Definition 2.1.1. The algebra A_β is a *PBW deformation* of $\Pi_0^{\otimes n}\#S_n$ if $\mathtt{gr}\,\mathsf{A}_\beta = \Pi_0^{\otimes n}\#S_n$.

Remark 2.1.2. When $n=1$ and Q is affine Dynkin of type ADE, any B-bilinear map $\beta : R \longrightarrow B$ gives a PBW deformation by [CBH, Cor. 3.6].

2.2 The first main result of this paper is the following.

Theorem 2.2.1. *Let $n>1$ and assume Q is an affine Dynkin quiver of type ADE. The algebra A_β is a PBW deformation of $\Pi_0^{\otimes n}\#S_n$ if and only if $\mathsf{A}_\beta = \mathsf{A}_{n,\lambda,\nu}$ for some $\lambda\in B$, $\nu\in\mathbb{C}$.*

Proof. When Q is of type A_0, the Theorem follows from [EG, Theorem 1.3]; see Remark 1.3.2. Hence, we may assume that Q has no edge-loop. Given any $x\in U$, write $\beta(x)$ in the form

$$\beta(x) = \sum \beta_\sigma^{j_1\cdots j_n}(x) e_{j_1}\otimes\cdots\otimes e_{j_n}\cdot\sigma$$

where $\beta_\sigma^{j_1\cdots j_n}(x)\in\mathbb{C}$ and the sum is taken over all vertices $j_1,\ldots,j_n\in I$ and permutations $\sigma\in S_n$.

First, we find the constraints on $\beta_\sigma^{j_1\cdots j_n}$ so that β is K-bilinear. The right S_n-linearity of β means that the $\beta_\sigma^{j_1\cdots j_n}$'s are determined by how they are defined on R. The left S_n-linearity of β means that for any $\sum k_1\otimes k_2\otimes\cdots\otimes k_n\in\mathsf{R}$ and $\tau\in S_n$, we have

$$\beta_\sigma^{j_1\cdots j_n}\big(\sum k_1\otimes k_2\otimes\cdots\otimes k_n\big) = \beta_{\tau\sigma\tau^{-1}}^{j_{\tau(1)}\cdots j_{\tau(n)}}\big(\sum k_{\tau(1)}\otimes k_{\tau(2)}\otimes\cdots\otimes k_{\tau(n)}\big)\,.$$

Consider any element $k_1\otimes\cdots\otimes k_n\in\mathsf{R}_\ell$ with $k_1,k_2,\ldots\in I$ except for $k_\ell = r_i\in R$. By the B-bilinearity of β, we must have $\beta_\sigma^{j_1\cdots j_n}(k_1\otimes\cdots\otimes k_n)=0$ if $j_1\neq k_1$, or $j_2\neq k_2,\ldots$, or $j_\ell\neq i,\ldots$ or $j_n\neq k_n$, or $k_{\sigma(p)}\neq k_p$ for any p.

Consider any element $\lfloor\varepsilon,\varepsilon'\rfloor$ with ε,ε' of the form (1.2.1)–(1.2.2). By the B-bilinearity of β, we must have $\beta_\sigma^{j_1\cdots j_n}(\lfloor\varepsilon,\varepsilon'\rfloor)=0$ if $j_1\neq i_1$, or $j_2\neq i_2,\ldots$, or $j_\ell\neq h(a)$ or $j_m\neq h(b)$, or $i_{\sigma(p)}\neq i_p$ for any p, or (ℓ,m in different cycles of σ), or (ℓ,m in same cycle of σ and $t(b)\neq h(a)$ or $t(a)\neq h(b)$). Note, in particular, that $\beta(\lfloor\varepsilon,\varepsilon'\rfloor)=0$ if $t(b)\neq h(a)$ or $t(a)\neq h(b)$.

Next, it is known that the B-algebra Π_0 is Koszul; this was proved in [Gr, Theorem 7.2] when Q is affine Dynkin of type A and in [MV, Theorem 1.9] when Q is affine Dynkin of type D or E. Thus, $\Pi_0^{\otimes n}\#S_n$ is a Koszul K-algebra, and so by [BG, Lemma 3.3], A_β

is a PBW deformation if and only if $\beta \otimes \mathtt{Id} = \mathtt{Id} \otimes \beta$ on $\big(\mathsf{R} \otimes_{\mathsf{B}} \mathsf{E}\big) \cap \big(\mathsf{E} \otimes_{\mathsf{B}} \mathsf{R}\big)$. Here, the equality takes place in M while the intersection takes place in $T_K^3 M$.

Observe that $\big(\mathsf{R} \otimes_{\mathsf{B}} \mathsf{E}\big) \cap \big(\mathsf{E} \otimes_{\mathsf{B}} \mathsf{R}\big)$ is spanned by the following two types of elements:

(1) For any $\varepsilon \in \mathsf{E}_\ell$, $\eta \in \mathsf{E}_m$ and $\zeta \in \mathsf{E}_r$ of the form

$$\varepsilon = e_{i_1} \otimes \cdots \otimes a \otimes \cdots \otimes h(b) \otimes \cdots \otimes h(c) \otimes \cdots \otimes e_{i_n} \,, \tag{2.2.2}$$

$$\eta = e_{i_1} \otimes \cdots \otimes t(a) \otimes \cdots \otimes b \otimes \cdots \otimes h(c) \otimes \cdots \otimes e_{i_n} \,, \tag{2.2.3}$$

$$\zeta = e_{i_1} \otimes \cdots \otimes t(a) \otimes \cdots \otimes t(b) \otimes \cdots \otimes c \otimes \cdots \otimes e_{i_n} \,, \tag{2.2.4}$$

where $a, b, c \in \overline{Q}$ and $i_1, \ldots, i_n \in I$, we have the element:

$$\lfloor \varepsilon, \eta \rfloor \zeta - \lfloor \varepsilon, \zeta \rfloor \eta + \lfloor \eta, \zeta \rfloor \varepsilon = \varepsilon \lfloor \eta, \zeta \rfloor - \eta \lfloor \varepsilon, \zeta \rfloor + \zeta \lfloor \varepsilon, \eta \rfloor \,.$$

Here, in the second term of the left hand side, the η is actually the η of (2.2.3) whose r-th entry is $t(c)$ instead of $h(c)$, and the ζ is actually the ζ of (2.2.4) whose m-th entry is $h(b)$ instead of $t(b)$. Throughout, we shall use similar convention.

(2) We use a similar convention as above. For any $x = \sum \varepsilon_i \eta_i \in \mathsf{R}_\ell$ (where $\varepsilon_i, \eta_i \in \mathsf{E}_\ell$) and $\zeta \in \mathsf{E}_m$, we have the element:

$$\big(\sum \varepsilon_i \eta_i\big)\zeta - \sum \lfloor \varepsilon_i, \zeta \rfloor \eta_i = \sum \varepsilon_i \lfloor \eta_i, \zeta \rfloor + \zeta\big(\sum \varepsilon_i \eta_i\big) \,.$$

We now see what the equation $\beta \otimes \mathtt{Id} = \mathtt{Id} \otimes \beta$ says when applied to elements of type (1). We have:

$$\beta(\lfloor \varepsilon, \eta \rfloor)\zeta - \beta(\lfloor \varepsilon, \zeta \rfloor)\eta + \beta(\lfloor \eta, \zeta \rfloor)\varepsilon = \varepsilon \beta(\lfloor \eta, \zeta \rfloor) - \eta \beta(\lfloor \varepsilon, \zeta \rfloor) + \zeta \beta(\lfloor \varepsilon, \eta \rfloor) \,.$$

Recall that this is an equality of elements in M. The edges which appear in $\beta(\lfloor \varepsilon, \eta \rfloor)\zeta$, $\beta(\lfloor \varepsilon, \zeta \rfloor)\eta$, and $\beta(\lfloor \eta, \zeta \rfloor)\varepsilon$ are, respectively, c, b, and a; the same for $\zeta\beta(\lfloor \varepsilon, \eta \rfloor)$, $\eta\beta(\lfloor \varepsilon, \zeta \rfloor)$, and $\varepsilon\beta(\lfloor \eta, \zeta \rfloor)$. It follows that for equality, we must have $\beta(\lfloor \varepsilon, \eta \rfloor)\zeta = \zeta\beta(\lfloor \varepsilon, \eta \rfloor)$ when $c \neq a, b$. Hence, we deduce that $\beta_\sigma^{j_1 \cdots j_n}(\lfloor \varepsilon, \eta \rfloor) = 0$ if $\sigma \neq s_{\ell m}$. Moreover, we also have

$$\beta_\sigma^{j_1 \cdots j_r \cdots j_n}(\lfloor \varepsilon, \eta \rfloor) = \beta_\sigma^{j_1 \cdots j'_r \cdots j_n}(\lfloor \varepsilon, \eta \rfloor)$$

where $r \neq \ell, m$ and $j_r = h(c)$, $j'_r = t(c)$ (for any edge $c \in \overline{Q}$). Here, by our convention, the r-th entry of $\lfloor \varepsilon, \eta \rfloor$ on the left hand side is $h(c)$ while the r-th entry of $\lfloor \varepsilon, \eta \rfloor$ on the right hand side is $t(c)$.

Now we see what the equation $\beta \otimes \mathtt{Id} = \mathtt{Id} \otimes \beta$ says when applied to elements of type (2). We get:

$$\beta\big(\sum \varepsilon_i \eta_i\big)\zeta - \sum \beta\big(\lfloor \varepsilon_i, \zeta \rfloor\big)\eta_i = \sum \varepsilon_i \beta\big(\lfloor \eta_i, \zeta \rfloor\big) + \zeta\beta\big(\sum \varepsilon_i \eta_i\big) \,. \tag{2.2.5}$$

Hence,

$$\beta_1^{j_1 \cdots j_m \cdots j_n}(x) = \beta_1^{j_1 \cdots j'_m \cdots j_n}(x)$$

where $m \neq \ell$ and $j_m = h(c)$, $j'_m = t(c)$ (for any edge $c \in \overline{Q}$). Here, by our convention, the m-th entry of x on the left hand side is $h(c)$ while the m-th entry of x on the right hand side is $t(c)$. Moreover, we have $\beta_\sigma^{j_1\cdots j_n}(x) = 0$ if σ is not 1 or $s_{\ell m}$ for any m.

Let $\sigma = s_{\ell m}$. Taking ζ to be ε' of (1.2.2) and x to be $e_{i_1} \otimes \cdots \otimes r_{i_\ell} \otimes \cdots \otimes e_{i_n}$, we deduce from (2.2.5) that $\beta_\sigma^{i_1\cdots i_n}(\lfloor \varepsilon, \varepsilon' \rfloor) = \beta_\sigma^{i_1\cdots i_n}(x)$ if $b \in Q$ and $a = b^*$; $\beta_\sigma^{i_1\cdots i_n}(\lfloor \varepsilon, \varepsilon' \rfloor) = -\beta_\sigma^{i_1\cdots i_n}(x)$ if $a \in Q$ and $b = a^*$; and $\beta_\sigma^{i_1\cdots i_n}(\lfloor \varepsilon, \varepsilon' \rfloor) = 0$ otherwise. $\square$

Remark 2.2.6. Let Q be a connected quiver. It is known from [Gr] and [MV] that if Q is not a finite Dynkin quiver, then the preprojective algebra Π_0 is Koszul. In this case, note that $(R \otimes_B E) \cap (E \otimes_B R) = 0$ when Q has more than one edge, and so by [BG], the deformed preprojective algebra Π_λ is PBW for any $\lambda \in B$; moreover, assuming furthermore that Q has no edge-loop, Theorem 2.2.1 is still true by same proof as above.

2.3 We end this section with some comments. First, $T_{\mathsf{B}}\mathsf{E}$ is the path algebra of the product quiver $\overline{Q} \times \cdots \times \overline{Q}$ whose vertex set is $I \times \cdots \times I$ and edge set is $\bigcup I \times \cdots \times \overline{Q} \times \cdots \times I$.

Next, let us consider the relations (ii) in Definition 1.2.3 for $n = 2$.

Example 2.3.1. When $n = 2$, the relations (ii) in Definition 1.2.3 means that, for any edge $a \in Q$:

$$(a^* \otimes h(a))(h(a) \otimes a) - (t(a) \otimes a)(a^* \otimes t(a)) = \nu \cdot (t(a) \otimes h(a))s_{12}\,;$$

and for any edges $a, b \in \overline{Q}$ with $a \neq b^*$ or $b \neq a^*$:

$$(a \otimes h(b))(t(a) \otimes b) - (h(a) \otimes b)(a \otimes t(b)) = 0\,.$$

3 Morita equivalence

3.1 We will give in the following lemma a more explicit presentation of the algebra $\mathsf{H}_{t,c}(\mathbf{\Gamma}_n)$ by generators and relations. Given any element $u \in L$ and $i \in [1, n]$, we will write $u_i \in V$ for u placed in the i-th factor L. Recall from §1.3 that $c = k \cdot 1 + \sum_{\gamma \in \Gamma \smallsetminus \{1\}} c'_\gamma \cdot \gamma \in \mathsf{Z}\Gamma$. From now on, we fix a basis $\{x, y\}$ of L with $\omega_L(x, y) = 1$.

Lemma 3.1.1. *The algebra $\mathsf{H}_{t,c}(\mathbf{\Gamma}_n)$ is the quotient of $TV \# \mathbf{\Gamma}_n$ by the following relations:*

(R1) *For any $i \in [1, n]$:*

$$[x_i, y_i] = t \cdot 1 + \frac{k}{2} \sum_{j \neq i} \sum_{\gamma \in \Gamma} s_{ij} \gamma_i \gamma_j^{-1} + \sum_{\gamma \in \Gamma \smallsetminus \{1\}} c'_\gamma \gamma_i\,.$$

(R2) *For any $u, v \in L$ and $i \neq j$:*

$$[u_i, v_j] = -\frac{k}{2} \sum_{\gamma \in \Gamma} \omega_L(\gamma u, v) s_{ij} \gamma_i \gamma_j^{-1}\,.$$

Proof. We first consider symplectic reflections of type (S). Let $s = s_{ij}\gamma_i\gamma_j^{-1}$. If $u \in V$, then $(u - su)/2 \in \texttt{Im}(\texttt{Id} - s)$ and $(u + su)/2 \in \texttt{Ker}(\texttt{Id} - s)$. Thus, for any $u, v \in V$, we have $\omega_s(u, v) = \omega(u - su, v - sv)/4 = \omega(u, v)/2 - \omega(u, sv)/2$. In particular,

$$\begin{array}{rcll} \omega_s(x_i, y_i) & = & 1/2\,, & \\ \omega_s(x_l, y_l) & = & 0 & \text{for } l \neq i, j\,, \\ \omega_s(u_i, v_j) & = & -\omega_L(u, \gamma^{-1}v)/2 & \text{for any } u, v \in L\ , \\ \omega_s(u_l, v_m) & = & 0 & \text{for any } u, v \in L,\ l \neq m, i, j\,. \end{array}$$

We next consider symplectic reflections of type (Γ). Let $s = \gamma_i$. Then $\texttt{Im}(\texttt{Id} - s) = \{u_i | u \in L\}$, and $\texttt{Ker}(\texttt{Id} - s)$ is spanned by u_j where $u \in L$, $j \neq i$. Thus,

$$\begin{array}{rcll} \omega_s(x_i, y_i) & = & 1\,, & \\ \omega_s(x_l, y_l) & = & 0 & \text{for } l \neq i\,, \\ \omega_s(u_l, v_m) & = & 0 & \text{for any } u, v \in L,\ l \neq m, i\ . \end{array}$$

□

3.2 Let us recall the classical McKay correspondence. Given the finite subgroup $\Gamma \subset Sp(L)$, we shall write its irreducible representations as N_i, $i \in I$. Consider the quiver with vertex set I and whose number of edges from i to j is the multiplicity of N_i in $L \otimes N_j$. This quiver is the double of an affine Dynkin quiver Q of type ADE. This construction gives a bijection between conjugacy classes of finite subgroups of $Sp(L)$ and affine Dynkin diagrams of type ADE.

3.3 Following [CB, §4] and [CBH, §3], we now define idempotent elements f_i and f in the group algebra $\mathbb{C}\Gamma$. For each $i \in I$, let δ_i be the dimension of the irreducible representation N_i. We fix an isomorphism $\mathbb{C}\Gamma \simeq \bigoplus_{i\in I} \mathrm{Mat}(\delta_i \times \delta_i)$. Let $E^i_{p,q}$ $(1 \le p, q \le \delta_i)$ be the element of $\mathbb{C}\Gamma$ with 1 in the (p, q)-entry of the matrix for the i-th summand and zero elsewhere. Let f_i be the idempotent $E^i_{1,1}$, and let $f = \sum_{i\in I} f_i$.

Note that, in the algebra $\mathbb{C}[\Gamma^n] = (\mathbb{C}\Gamma)^{\otimes n}$, we have

$$f^{\otimes n} = \sum_{i_1,\ldots,i_n \in I} f_{i_1} \otimes \cdots \otimes f_{i_n}$$

and

$$\begin{aligned} &\sum_{i_1,p_1,\ldots,i_n,p_n} (E^{i_1}_{p_1,1} \otimes \cdots \otimes E^{i_n}_{p_n,1}) f^{\otimes n} (E^{i_1}_{1,p_1} \otimes \cdots \otimes E^{i_n}_{1,p_n}) \\ = &\sum_{i_1,p_1,\ldots,i_n,p_n} E^{i_1}_{p_1,1}E^{i_1}_{1,p_1} \otimes \cdots \otimes E^{i_n}_{p_n,1}E^{i_n}_{1,p_n} \quad = 1^{\otimes n}\,. \end{aligned} \tag{3.3.1}$$

3.4 We state here some observations which we will use later. First, we have an isomorphism

$$\mathsf{B} \xrightarrow{\sim} f^{\otimes n}\mathbb{C}[\Gamma^n]f^{\otimes n} = \bigoplus_{i_1,\dots,i_n\in I} \mathbb{C}\cdot f_{i_1}\otimes\cdots\otimes f_{i_n} \tag{3.4.1}$$

defined by

$$e_{i_1}\otimes\cdots\otimes e_{i_n} \mapsto f_{i_1}\otimes\cdots\otimes f_{i_n}\,.$$

Now $V\otimes\mathbb{C}[\Gamma^n]$ is a $\mathbb{C}[\Gamma^n]$-bimodule, where the left action is the diagonal one. We have:

$$\begin{aligned}
&f^{\otimes n}(V\otimes\mathbb{C}[\Gamma^n])f^{\otimes n}\\
&= \bigoplus_{i_1,\dots,j_n} (f_{i_1}\otimes\cdots\otimes f_{i_n})(L^{\oplus n}\otimes \underbrace{\mathbb{C}\Gamma\otimes\cdots\otimes\mathbb{C}\Gamma}_{n})(f_{j_1}\otimes\cdots\otimes f_{j_n})\\
&= \bigoplus_{l=1}^{n}\bigoplus_{i_1,\dots,j_n} \mathtt{Hom}_\Gamma(N_{i_1},N_{j_1})\otimes\cdots\otimes\mathtt{Hom}_\Gamma(N_{i_l},L\otimes N_{j_l})\otimes\cdots\otimes\mathtt{Hom}_\Gamma(N_{i_n},N_{j_n})\\
&\simeq \mathsf{E}\,.
\end{aligned} \tag{3.4.2}$$

It follows from (3.4.1)–(3.4.2) that

$$f^{\otimes n}T_{\mathbb{C}[\Gamma^n]}(V\otimes\mathbb{C}[\Gamma^n])f^{\otimes n} \simeq T_{\mathsf{B}}\mathsf{E} \tag{3.4.3}$$

and

$$f^{\otimes n}(TV\#\mathbf{\Gamma}_n)f^{\otimes n} = f^{\otimes n}(T_{\mathbb{C}[\Gamma^n]}(V\otimes\mathbb{C}[\Gamma^n])\#S_n)f^{\otimes n} \simeq T_{\mathsf{B}}\mathsf{E}\,\#S_n\,. \tag{3.4.4}$$

3.5 By (3.3.1), the algebra $\mathsf{H}_{t,c}(\mathbf{\Gamma}_n)$ is Morita equivalent to the algebra $f^{\otimes n}\mathsf{H}_{t,c}(\mathbf{\Gamma}_n)f^{\otimes n}$. By (3.4.4), $f^{\otimes n}\mathsf{H}_{t,c}(\mathbf{\Gamma}_n)f^{\otimes n}$ is isomorphic to a quotient of $T_{\mathsf{B}}\mathsf{E}\,\#S_n$.

Remark 3.5.1. When $n=1$, there is no parameter k and $\mathsf{H}_{t,c}(\mathbf{\Gamma}_n)$ is the algebra denoted by $\mathscr{S}^\lambda$ in [CBH], with λ_i being the trace of $t\cdot 1+\sum_{\gamma\neq 1}c'_\gamma\gamma$ on N_i; cf. Lemma 3.1.1. In this case, it was proved in [CBH, Theorem 3.4] that $f\mathscr{S}^\lambda f\simeq\Pi_\lambda$.

The second main result of this paper is the following generalization of [CBH, Theorem 3.4]; cf. also [CB, Theorem 4.12].

Theorem 3.5.2. *When $n>1$, there is an isomorphism $f^{\otimes n}\mathsf{H}_{t,c}(\mathbf{\Gamma}_n)f^{\otimes n}\simeq\mathsf{A}_{n,\lambda,\nu}$, where λ_i is the trace of $t\cdot 1+\sum_{\gamma\neq 1}c'_\gamma\gamma$ on N_i and $\nu=\frac{k|\Gamma|}{2}$.*

Proof. When $\Gamma=\{1\}$, the Theorem is trivial by Remark 1.3.2. Hence, we may assume $\Gamma\neq\{1\}$, and so Q is not of type A_0.

Denote by $\zeta:\mathbb{C}\to L\otimes L$ the linear map that sends 1 to $y\otimes x-x\otimes y$. By Lemma 3.2 of [CBH] and its proof, for each edge $a\in Q$, there are Γ-equivariant monomorphisms

$$\theta_a: N_{t(a)}\longrightarrow L\otimes N_{h(a)} \qquad\text{and}\qquad \phi_a: N_{h(a)}\longrightarrow L\otimes N_{t(a)}$$

such that for each vertex i, we have

$$\sum_{a\in Q,h(a)=i}(\mathtt{Id}_L\otimes\theta_a)\phi_a-\sum_{a\in Q,t(a)=i}(\mathtt{Id}_L\otimes\phi_a)\theta_a=-\delta_i(\zeta\otimes\mathtt{Id}_{N_i})$$

as maps from N_i to $L\otimes L\otimes N_i$, and such that

$$(\omega_L\otimes\mathtt{Id}_{N_{t(a)}})(\mathtt{Id}_L\otimes\phi_a)\theta_a=-\delta_{h(a)}\mathtt{Id}_{N_{t(a)}}$$

and

$$(\omega_L\otimes\mathtt{Id}_{N_{h(a)}})(\mathtt{Id}_L\otimes\theta_a)\phi_a=\delta_{t(a)}\mathtt{Id}_{N_{h(a)}}\,.$$

Moreover, the θ_a,ϕ_a ($a\in Q$) combine to give a basis for each of the spaces $\mathtt{Hom}_\Gamma(N_i,L\otimes N_j)$. By (3.4.1)–(3.4.4), we have an isomorphism $T_{\mathsf{B}}\mathsf{E}\#S_n\ \xrightarrow{\sim}\ f^{\otimes n}(TV\#\mathbf{\Gamma}_n)f^{\otimes n}$ such that

$$e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_n}\cdot\sigma\mapsto f_{i_1}\otimes f_{i_2}\otimes\cdots\otimes f_{i_n}\cdot\sigma\,,$$

$$e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes a\otimes\cdots\otimes e_{i_n}\cdot\sigma\mapsto f_{i_1}\otimes f_{i_2}\otimes\cdots\otimes\phi_a\otimes\cdots\otimes f_{i_n}\cdot\sigma\,,$$

$$e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes a^*\otimes\cdots\otimes e_{i_n}\cdot\sigma\mapsto f_{i_1}\otimes f_{i_2}\otimes\cdots\otimes\theta_a\otimes\cdots\otimes f_{i_n}\cdot\sigma\,,$$

for any $i_1,\dots,i_n\in I$, $a\in Q$, and $\sigma\in S_n$.

Denote by J the subspace of $TV\,\#\mathbf{\Gamma}_n$ spanned by elements of the form $[u,v]-\kappa(u,v)$ with $u,v\in V$. The algebra $\mathsf{H}_{t,c}(\mathbf{\Gamma}_n)$ is the quotient of $TV\,\#\mathbf{\Gamma}_n$ by the two sided ideal generated by J. Thus, $f^{\otimes n}\mathsf{H}_{t,c}(\mathbf{\Gamma}_n)f^{\otimes n}$ is the quotient of $f^{\otimes n}(TV\,\#\mathbf{\Gamma}_n)f^{\otimes n}$ by the ideal $f^{\otimes n}(TV\#\,\mathbf{\Gamma}_n)J(TV\#\,\mathbf{\Gamma}_n)f^{\otimes n}$. By (3.3.1), we have

$$f^{\otimes n}(TV\#\,\mathbf{\Gamma}_n)J(TV\#\,\mathbf{\Gamma}_n)f^{\otimes n}=f^{\otimes n}(TV\#\,\mathbf{\Gamma}_n)f^{\otimes n}\mathbb{C}[\Gamma^n]J\mathbb{C}[\Gamma^n]f^{\otimes n}(TV\#\,\mathbf{\Gamma}_n)f^{\otimes n}\,.$$

Hence, $f^{\otimes n}\mathsf{H}_{t,c}(\mathbf{\Gamma}_n)f^{\otimes n}$ is the quotient of $f^{\otimes n}(TV\,\#\mathbf{\Gamma}_n)f^{\otimes n}$ by the two sided ideal generated by $f^{\otimes n}\mathbb{C}[\Gamma^n]J\mathbb{C}[\Gamma^n]f^{\otimes n}$. We will show that, via the isomorphism $T_{\mathsf{B}}\mathsf{E}\#S_n\simeq f^{\otimes n}(TV\#\,\mathbf{\Gamma}_n)f^{\otimes n}$ constructed above, the two sided ideal that $f^{\otimes n}\mathbb{C}[\Gamma^n]J\mathbb{C}[\Gamma^n]f^{\otimes n}$ generates gives precisely the relations that define $\mathsf{A}_{n,\lambda,\nu}$ as a quotient of $T_{\mathsf{B}}\mathsf{E}\#S_n$. We will use the description of J given by Lemma 3.1.1.

First, we consider the relations (R1) in Lemma 3.1.1. Observe that for any $g\in\Gamma$, since ω_L is Γ-invariant, we have

$$g(x\otimes y-y\otimes x)=(x\otimes y-y\otimes x)g\ \in TL\,\#\Gamma\,,$$

and since $c\in\mathsf{Z}\Gamma$, we have

$$g(t\cdot 1+\sum_{\gamma\neq 1}c'_\gamma\gamma)=(t\cdot 1+\sum_{\gamma\neq 1}c'_\gamma\gamma)g\ \in\mathbb{C}\Gamma\,.$$

Also, for any $g,h\in\Gamma$,

$$g_ih_j\big(\sum_{\gamma\in\Gamma}s_{ij}\gamma_i\gamma_j^{-1}\big)=\sum_{\gamma\in\Gamma}s_{ij}(h\gamma)_i(\gamma g^{-1})_j^{-1}=\big(\sum_{\gamma\in\Gamma}s_{ij}\gamma_i\gamma_j^{-1}\big)g_ih_j\ \in\mathbb{C}[\mathbf{\Gamma}_n]\,.$$

Hence, if $J_1 \subset J$ is spanned by elements of type (R1), then

$$f^{\otimes n}\mathbb{C}[\Gamma^n]J_1\mathbb{C}[\Gamma^n]f^{\otimes n} = f^{\otimes n}J_1 f^{\otimes n}\mathbb{C}[\Gamma^n]f^{\otimes n}\,.$$

For any $i_1, \dots, i_n \in I$ and $\ell \in [1, n]$, we have

$$\begin{aligned} f_{i_1} \otimes \cdots \otimes f_{i_n} \cdot [x_\ell, y_\ell] &= [x_\ell, y_\ell] \cdot f_{i_1} \otimes \cdots \otimes f_{i_n} \\ &= f_{i_1} \otimes \cdots \otimes \frac{1}{\delta_{i_\ell}}\Big(\sum_{a \in Q, h(a)=i_\ell} \phi_a\theta_a - \sum_{a\in Q, t(a)=i_\ell} \theta_a\phi_a\Big) \otimes \cdots \otimes f_{i_n} \end{aligned}$$

and

$$f_{i_1} \otimes \cdots \otimes f_{i_n}(t \cdot 1 + \sum_{\gamma\neq 1} c'_\gamma \gamma_\ell) = \frac{1}{\delta_{i_\ell}} \lambda_{i_\ell} f_{i_1} \otimes \cdots \otimes f_{i_n}\,.$$

Moreover, keeping in mind that we have fixed an isomorphism $\mathbb{C}\Gamma \simeq \bigoplus_{i\in I} \mathrm{Mat}(\delta_i \times \delta_i)$, we have by orthogonality relations for matrix coefficients (see [Se, p.14])

$$f_{i_1} \otimes \cdots \otimes f_{i_n} \cdot (\sum_\gamma s_{\ell j}\gamma_\ell\gamma_j^{-1}) \cdot f_{i_1} \otimes \cdots \otimes f_{i_n} = \begin{cases} \frac{|\Gamma|}{\delta_{i_\ell}} f_{i_1} \otimes \cdots \otimes f_{i_n} & \text{if } i_j = i_\ell\,, \\ 0 & \text{else}\,. \end{cases}$$

(Note that $f_i\gamma f_i$ (where $i \in I, \gamma \in \Gamma$) is equal to f_i times the corresponding matrix coefficient for the action of γ on N_i.) Hence, (R1) gives the relations (i) of Definition 1.2.3.

Next, we find the relations that come from (R2) in Lemma 3.1.1. To ease notations, we will assume without loss of generality that $n = 2$. (See Example 2.3.1.)

For any $u, v \in L$ and $g, h \in \Gamma$, note that

$$(g \otimes h) \cdot [u_1, v_2] = [(gu)_1, (hv)_2] \cdot (g \otimes h)\,,$$

and

$$\begin{aligned} (g \otimes h) \cdot (\sum_\gamma \omega_L(\gamma u, v)s_{12}\gamma_1\gamma_2^{-1}) &= \sum_\gamma \omega_L(\gamma u, v)s_{12}(h\gamma)_1(\gamma g^{-1})_2^{-1} \\ &= (\sum_\gamma \omega_L(\gamma g u, h v)s_{12}\gamma_1\gamma_2^{-1}) \cdot (g \otimes h)\,. \end{aligned}$$

Now, for any $i, j, k, l \in I$, we have

$$\begin{aligned} &(f_i \otimes f_j)(g \otimes 1)[u_1, v_2](1 \otimes h)(f_k \otimes f_l) \\ &= \sum_{i_1,p_1,i_2,p_2} (f_i g \otimes f_j)\big(u_1 \otimes (E^{i_1}_{p_1,1} \otimes E^{i_2}_{p_2,1})\big) \bigotimes (E^{i_1}_{1,p_1} \otimes E^{i_2}_{1,p_2})\big(v_2 \otimes (f_k \otimes hf_l)\big) \\ &\quad - \sum_{i_1,p_1,i_2,p_2} (f_i g \otimes f_j)\big(v_2 \otimes (g^{-1}E^{i_1}_{p_1,1} \otimes hE^{i_2}_{p_2,1})\big) \bigotimes (E^{i_1}_{1,p_1} g \otimes E^{i_2}_{1,p_2}h^{-1})\big(u_1 \otimes (f_k \otimes hf_l)\big) \\ &= (f_i g \otimes f_j)\big(u_1 \otimes (f_k \otimes f_j)\big) \bigotimes (f_k \otimes f_j)\big(v_2 \otimes (f_k \otimes hf_l)\big) \\ &\quad - (f_i \otimes f_j)\big(v_2 \otimes (f_i \otimes hf_l)\big) \bigotimes (f_i g \otimes f_l)\big(u_1 \otimes (f_k \otimes f_l)\big) \end{aligned} \tag{3.5.3}$$

Note that via our identifications in §3.4, the last line of (3.5.3) is an element of $T^2_{\mathsf{B}}\mathsf{E} = \mathsf{E}\bigotimes_{\mathsf{B}}\mathsf{E}$. Now, on the other hand,

$$\begin{aligned}(f_i\otimes f_j)(g\otimes 1)\Big(\sum_\gamma \omega_L(\gamma u,v)s_{12}\gamma_1\gamma_2^{-1}\Big)(1\otimes h)(f_k\otimes f_l)\\ = s_{12}\sum_\gamma \omega_L(\gamma u,v)(f_j\gamma f_k)\otimes(f_i g\gamma^{-1}hf_l)\,.\end{aligned} \tag{3.5.4}$$

Observe that for any edge $a\in\overline{Q}$, we can find $g_a,h_a\in\Gamma$ and $u_a,v_a\in L$ such that

$$f_{t(a)}g_a(u_a\otimes f_{h(a)})\neq 0 \qquad\text{and}\qquad f_{h(a)}(v_a\otimes h_a f_{t(a)})\neq 0\,.$$

Suppose that Q is not of type A_1, so that each of the spaces $f_i(L\otimes\mathbb{C}\Gamma)f_j$ is at most one dimensional. For any $i,j\in I$, there is a bijection

$$(f_i\mathbb{C}\Gamma\otimes L\otimes\mathbb{C}\Gamma f_j)^\Gamma\to f_i(L\otimes\mathbb{C}\Gamma)f_j : \alpha\otimes u\otimes\beta\mapsto\alpha(u\otimes\beta)\,.$$

Here, the action of $\gamma\in\Gamma$ on $\alpha\otimes u\otimes\beta\in f_i\mathbb{C}\Gamma\otimes L\otimes\mathbb{C}\Gamma f_j$ is $\alpha\gamma^{-1}\otimes\gamma u\otimes\gamma\beta$. There is also a Γ-equivariant non-degenerate pairing

$$(f_i\mathbb{C}\Gamma\otimes L\otimes\mathbb{C}\Gamma f_j)\bigotimes(f_j\mathbb{C}\Gamma\otimes L\otimes\mathbb{C}\Gamma f_i)\to\mathbb{C}$$

$$(\alpha\otimes u\otimes\beta)\bigotimes(\alpha'\otimes u'\otimes\beta')\mapsto(\alpha\beta')(\alpha'\beta)\omega_L(u,u')$$

Thus, for any edge $a\in\overline{Q}$, we may assume that $\omega_L(u_a,v_a)=1$. Moreover, $f_{t(a)}g(v_a\otimes f_{h(a)})=0$ if $f_{h(a)}(v_a\otimes hf_{t(a)})\neq 0$; and $f_{h(a)}(u_a\otimes hf_{t(a)})=0$ if $f_{t(a)}g(u_a\otimes f_{h(a)})\neq 0$.

Note that if $k\neq j$ or $l\neq i$, then the expression in (3.5.4) is zero. Hence, if $a:k\to i$ and $b:l\to j$ are two edges of $\overline{Q}$ such that $b\neq a^*$ or $a\neq b^*$, then we obtain from (3.5.3) and (R2) that

$$(a\otimes h(b))(t(a)\otimes b)-(h(a)\otimes b)(a\otimes t(b))=0\,.$$

Now suppose $k=j$ and $l=i$. If N_i is not an irreducible component of $L\otimes N_j$, then the expression in (3.5.4) is zero by orthogonality of matrix coefficients on N_i and $L\otimes N_j$; in this case, note that (3.5.3) is also zero. Thus, suppose there is an edge $a:i\to j$ in $\overline{Q}$. We consider the case that $a\in Q$; the case $a\notin Q$ is completely similar. We have $N_i\subset L\otimes N_j$ via θ_a. Hence, we have a decomposition into irreducible components $L\otimes N_j=N_i\bigoplus\cdots$. A basis for $N_i\bigoplus\cdots$ compatible with this direct sum decomposition is

$$\xi_1:=f_i=E^i_{1,1}\,,\quad \xi_2:=E^i_{2,1}\,,\quad \xi_3:=E^i_{3,1}\,,\quad\ldots\,,$$

and a basis for $L\otimes N_j$ is

$$\mu_1:=u_a\otimes f_j=u_a\otimes E^j_{1,1}\,,\quad \mu_2:=u_a\otimes E^j_{2,1}\,,\quad\ldots\,,\quad \mu_{2\delta_j}:=v_a\otimes E^j_{\delta_j,1}\,.$$

Define the matrix $\tau=(\tau_{p,q})$ by $\mu_q=\sum_p\tau_{p,q}g^{-1}\xi_p$ and the matrix $\varrho=(\varrho_{p,q})$ by $h\xi_q=\sum_p\varrho_{p,q}\mu_p$.

Using the fact that the composition

$$N_j \overset{\phi_a}{\hookrightarrow} L \otimes N_i \overset{\theta_a}{\hookrightarrow} L \otimes L \otimes N_j \overset{\omega_L \otimes 1}{\longrightarrow} N_j$$

is multiplication by δ_i, we get

$$f_i g(u_a \otimes f_j) = \tau_{1,1} f_i \qquad \text{and} \qquad f_j(v_a \otimes h f_i) = -\frac{\varrho_{1,1}}{\delta_i} f_j\,.$$

Now consider the matrix coefficients for $\gamma : L \otimes N_j \longrightarrow L \otimes N_j$, where the matrix representing γ is taken with respect to the basis $\{g^{-1}\xi_p\}_{p=1,2,\dots}$ for the domain of γ and the basis $\{h\xi_p\}_{p=1,2,\dots}$ for the image of γ, vice versa for the matrix representing γ^{-1}. Since each irreducible representation of Γ appears in $L \otimes N_j$ at most once, and the bases $\{g^{-1}\xi_p\}_{p=1,2,\dots}$ and $\{h\xi_p\}_{p=1,2,\dots}$ respect the decomposition into irreducible components, the usual orthogonality relations, proved for example in [Se, p.14], continue to hold in this situation, that is, we have

$$\sum_{\gamma \in \Gamma} \omega_L(\gamma u_a, v_a)(f_j \gamma f_j) \otimes (f_i g \gamma^{-1} h f_i) = \frac{|\Gamma| \tau_{1,1} \varrho_{1,1}}{\delta_i} f_j \otimes f_i\,.$$

Hence, taking $u = u_a$ and $v = v_a$ in (3.5.3)–(3.5.4), we get from (R2) the relation:

$$(a^* \otimes h(a))(h(a) \otimes a) - (t(a) \otimes a)(a^* \otimes t(a)) = s_{12} \frac{k|\Gamma|}{2} (h(a) \otimes t(a))\,.$$

Note that $\tau_{1,1}$ and $\varrho_{1,1}$ are non-zero when $g = g_a$ and $h = h_a$.

Now take $u = u_a$ and $v = u_a$ in (3.5.3)–(3.5.4). If $f_i g(u_a \otimes f_j) \neq 0$, then $f_j(u_a \otimes h f_i) = 0$ and so $\varrho_{\delta_j+1,1} = 0$. Thus, both sides of the relation that (R2) gives in this case are zero. Similarly if $u = v_a$ and $v = v_a$.

When Q is of type A_1, $\Gamma = \mathbb{Z}/2\mathbb{Z}$, and in particular it is abelian. It is straightforward to check the relations directly in this case.

We conclude that (R2) gives the relations (ii) of Definition 1.2.3. □

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Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA;
`wlgan@math.mit.edu`

Department of Mathematics, University of Chicago, Chicago, IL 60637, USA;
`ginzburg@math.uchicago.edu`